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GREEN'S TENSOR FOR AN ELASTIC CYLINDER AND ITS APPLICATIONS IN THE DEVELOPMENT OF THE SAINT-VENANT THEORY†

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Using earlier results [1, 2], Green's tensor for an elastic cylinder is constructed. This enables the case of multiple eigenvalues to be considered. The static problem receives special attention. Green's tensor for an infinite cylinder is constructed as an expansion in terms of elementary solutions. For a cylinder of finite length the construction of Green's tensor can be reduced to an infinite system. An asymptotic analysis of the classical Saint-Venant problems is carried out. The notions of the vector-valued Green's function and Green's tensor are introduced into the Saint-Venant theory and they are constructed in explicit form.

The solution of the problem of the steady-state vibrations of an infinite cylinder subject to a point force applied at an arbitrary point was represented in [3, 4] in the form of expansions in terms of homogeneous elementary solutions and can be regarded as Green's tensor of the boundary-value problem. However, the case when the eigenvalue problem on the cross-section has multiple spectrum points has not been considered. The case of critical frequencies and the very important case of the static problem have thereby been excluded from consideration because the null eigenvalue of the latter has an algebraic multiplicity of two. In particular, the null eigenvalue corresponds to the classical solutions of the Saint-Venant problem [5] on the stretching, twisting, and bending of a cylinder of finite length by forces applied to one of its ends.

1. Let $V = S \times [-\infty, \infty]$ be a domain occupied by an ideal elastic medium, let S be the cross-section of the cylinder, ∂S the boundary of S, and let Γ be the lateral surface of the cylinder. We place the origin of a Cartesian system of coordinates $x_1x_2x_3$ on the principal axes of the section S_0, the x_3 axis being parallel to the generatrix of the cylinder.

We shall consider the harmonic vibrations of the cylinder generated by a point force proportional to $e^{-i\omega t}$ and applied at a point O' with coordinates $(x_1', x_2', 0)$.

We shall assume that the lateral surface of the cylinder is stress-free.

The following notation is used below: $\mathbf{u} = \{u_k\}_{k=1}^3$ is the displacement vector, $\boldsymbol{\sigma} = \{\sigma_{3k}\}_{k=1}^3$ is the stress vector in the planes orthogonal to the x_3 axis, $\mathbf{F}_0 = \mathbf{F}\delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3')$ is a vector representing the point force, $\mathbf{F} = \{F_k\}_{k=1}^3$, $\delta(x)$ is the delta-function, and $\mathbf{w} = \{\boldsymbol{\mu}, \boldsymbol{\sigma}\}$ is an extended six-component vector. Henceforth, $\mathbf{u}$ and $\boldsymbol{\sigma}$ will be called the u- and σ-components.

We shall consider $\mathbf{w}$ to be a vector-valued function $\mathbf{w}(x)(x = x_3)$ with values in the Hilbert space $H' = H \oplus H$ with scalar product

$$(\mathbf{w}^{(1)}, \mathbf{w}^{(2)})_{H'} = (\mathbf{u}^{(1)}, \mathbf{u}^{(2)})_H + (\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)})_H$$

$$(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})_H = \int_S \mathbf{u}^{(1)} \cdot \bar{\mathbf{u}}^{(2)} ds = \int_S \sum_{k=1}^3 u_k^{(1)} \bar{u}_k^{(2)} ds$$

The equations of equilibrium and of steady-state vibrations together with the boundary conditions on Γ can be written in the form [6]

$$d\mathbf{w}/dx - iT\mathbf{w} = \mathbf{K} \tag{1.1}$$

where T is an unbounded operator in H' and $\mathbf{K} = i\{0; \mathbf{F}_0\}$.

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The eigenvalue problem

$$T\mathbf{v} = \gamma\mathbf{v} \tag{1.2}$$

corresponds to the homogeneous problem (1.1).

We know [2, 7] that for any bounded frequency ω the spectrum of T can be split into $\Lambda = \Lambda^0 \cup \Lambda^+ \cup \Lambda^-$, where Λ^0 is a finite set of real eigenvalues (EVs) γ_r, and Λ^+, Λ^- are unbounded sets of complex EVs γ_k^+, γ_k^-($\mathrm{Im}\,\gamma_k^+ > 0$, $\mathrm{Im}\,\gamma_k^- < 0$). A multiple EV corresponds to the case $\omega'(\gamma_k) = 0$, the group velocity being equal to zero.

We will denote by W the set of elementary solutions of the homogeneous equation (1.1). Each element of this set can either be represented in the form

$$\mathbf{w}_s = \mathbf{v}_s e^{i\gamma_s x}$$

where γ_s is a simple EV of the eigenvalue problem (1.2), or in the form

$$\mathbf{w}_{sn} = e^{i\gamma_s x} \sum_{l=0}^{n} \frac{(ix)^n}{l!} \mathbf{v}_{s,n-l}$$

if γ_s is a multiple EV, which may correspond to one or more systems of eigenvectors and associated vectors (Jordan chains) $\mathbf{v}_{t0}, \mathbf{v}_{t1}, \dots, \mathbf{v}_{tNt}$.

It has been established [1, 2] that, for any structure of the spectrum, $W = W^+ \cup W^-$, where W^+ and W^- are defined as follows: in the case of real EVs ($s = r$)$\mathbf{w}_r \in W^+$, if $(J\mathbf{w}_r, \mathbf{w}_r) > 0$, and $\mathbf{w}_r \in W^-$ if $(J\mathbf{w}_r, \mathbf{w}_r) < 0$, where $J = i\| J_{kl} \|$, $J_{14} = J_{25} = J_{36} = -1$, $J_{41} = J_{52} = J_{63} = 1$ with $J_{kl} = 0$ for the remaining ones; in the case of complex EVs ($s = k$)$\mathbf{w}_k \in W^+$ if $\gamma_k = \gamma_k^+$, and $\mathbf{w}_k \in W^-$ if $\gamma_k = \gamma_k^-$. Below $\mathbf{w}_s = \mathbf{w}_s^+(\mathbf{w}_s^-)$ if $\mathbf{w}_s \in W^+$ (W^-).

Since the energy flow through the cross-section of the cylinder is

$$P(\mathbf{w}) = \omega[\mathbf{w}, \mathbf{w}]/4$$

$$[\mathbf{w}, \mathbf{w}] = (J\mathbf{w}, \mathbf{w})_{H'} = i[(\mathbf{u}, \boldsymbol{\sigma})_H - \overline{(\mathbf{u}, \boldsymbol{\sigma})_H}]$$

it follows that, in the case of oscillations, W is decomposed in accordance with the energy radiation principle [8].

We denote by $\mathbf{v}_s = \mathbf{w}_s(0) = \{\mathbf{a}_s, \mathbf{b}_s\}$ the traces of the elementary solutions in the section $x = 0$. We will use the following biorthogonality properties below

$$[\mathbf{v}_r^{\pm}, \mathbf{v}_l^{\pm}] = \pm 2p_r\delta_{rl}, \quad [\mathbf{v}_r^{\pm}, \mathbf{v}_l^{\mp}] = 0 \tag{1.3}$$

$$[\mathbf{v}_k^{\pm}, \mathbf{v}_m^{\mp}] = 2p_k^{\pm}\delta_{km}, \quad p_k^- = -\bar{p}_k^+ \tag{1.4}$$

as well as the properties of completeness and minimality in H', which follow from [2, 9, 10]. In (1.3) and (1.4) the subscripts r and l refer to vectors corresponding to real EVs, while m and k refer to those corresponding to complex EVs.

We will denote the solution of the problem under consideration by

$$\mathbf{G}(x) = \mathbf{G}(x_1, x_2, x_3, x_1', x_2', 0)$$

and call it the vector-valued Green's function. We shall seek it in the form

$$\begin{aligned} \mathbf{G}(x) &= \mathbf{G}^+(x) = \sum_s C_s^+ \mathbf{w}_s^+, \quad x > 0 \\ \mathbf{G}(x) &= \mathbf{G}^-(x) = \sum_s C_s^- \mathbf{w}_s^-, \quad x < 0 \end{aligned} \tag{1.5}$$

Since the right-hand side of (1.1) is proportional to $\delta(x)$ (see (1.2)), the solution undergoes a jump at the cross-section $x = 0$, so that

$$\mathbf{G}(+0) - \mathbf{G}(-0) = \mathbf{G}^+(0) - \mathbf{G}^-(0) = \mathbf{K}_0 \tag{1.6}$$

$$\mathbf{K}_0 = i\{0, \mathbf{P}_0\}, \quad \mathbf{P}_0 = \mathbf{F}\delta(x_1 - x_1')\delta(x_2 - x_2')$$

Substituting (1.5) into (1.6) and using the biorthogonality properties (1.3) and (1.4), we obtain in the case of real EVs

$$C_r^{\pm} = i(2p_r)^{-1}\bar{\mathbf{a}}_r^{\pm} \cdot \mathbf{F} \tag{1.7}$$

in the case of complex EVs

$$C_k^{\pm} = \pm i(2p_k^{\pm})^{-1}\bar{\mathbf{a}}_r^{\mp} \cdot \mathbf{F} \tag{1.8}$$

In (1.7) and (1.8) $\mathbf{a}_s^{\pm} = \mathbf{u}_s^{\pm}(x_1', x_2', 0)$ are the components of the displacement vector of the corresponding elementary solution at O'.

Substituting (1.7) and (1.8) into (1.5), we get

$$\begin{gathered}\mathbf{G}^{\pm}(x) = \mathbf{G}_0^{\pm}(x) + \mathbf{G}_1^{\pm}(x), \quad \mathbf{G}_0^{\pm}(x) = i\sum_r (2p_r)^{-1}(\bar{\mathbf{a}}_r^{\pm} \cdot \mathbf{F})\mathbf{w}_r^{\pm}(x) \\ \mathbf{G}_1^{\pm}(x) = \pm i\sum_k (2p_k^{\pm})^{-1}(\bar{\mathbf{a}}_k^{\mp} \cdot \mathbf{F})\mathbf{w}_k^{\pm}(x)\end{gathered} \tag{1.9}$$

Taking the u-component, we obtain expressions for the components of Green's tensor

$$u_{ml} = i\sum_r (2p_r)^{-1}\bar{a}_{rl}^{\pm}u_{rm}^{\pm} \pm i\sum_k (2p_k^{\pm})^{-1}\bar{a}_{kl}^{\mp}u_{km}^{\pm}$$

2. Consider the static problem ($\omega = 0$). In this case the real part of the spectrum of (1.2) consists of a 12-fold EV $\gamma = 0$. The corresponding invariant subspace of T is defined by the following system of eigenvectors and associated vectors $\mathbf{v}_r = \{\mathbf{a}_r, \mathbf{b}_r\}$ ($r = \overline{1, 12}$)

$$\begin{gathered}\mathbf{a}_1 = (1,0,0), \quad \mathbf{a}_2 = (0,0,-i\xi_1), \quad \mathbf{a}_3 = (0,1,0) \\ \mathbf{a}_4 = (0,0,-i\xi_2), \quad \mathbf{a}_5 = (0,0,1), \quad \mathbf{a}_6 = (-\xi_2 + \xi_2^*, \xi_1 - \xi_1^*, 0) \\ \mathbf{a}_7 = (0,0,i\theta_1), \quad \mathbf{a}_8 = (\psi_{11}, \psi_{12}, 0), \quad \mathbf{a}_9 = (0,0,i\theta_2) \\ \mathbf{a}_{10} = (\psi_{21}, \psi_{22}, 0), \quad \mathbf{a}_{11} = (-i\nu\xi_1, -i\nu\xi_2, 0), \quad \mathbf{a}_{12} = (0,0,i\theta) \\ \mathbf{b}_r = (0,0,0) \quad (r = \overline{1,6}) \\ \mathbf{b}_7 = i(\tau_{11}, \tau_{12}, 0), \quad \mathbf{b}_8 = (0,0,-E_0\xi_1), \quad \mathbf{b}_9 = i(\tau_{21}, \tau_{22}, 0) \\ \mathbf{b}_{10} = (0,0,-E_0\xi_2), \quad \mathbf{b}_{11} = (0,0,iE_0), \quad \mathbf{b}_{12} = i(\partial_2\Phi, -\partial_1\Phi, 0)\end{gathered} \tag{2.1}$$

To each vector $\mathbf{v}_r$ there corresponds an elementary solution

$$\begin{gathered}\mathbf{w}_r(\xi) = \{\mathbf{u}_r(\xi), \boldsymbol{\sigma}_r(\xi)\}: \\ \mathbf{u}_1 = \mathbf{a}_1, \quad \mathbf{u}_2 = i\xi\mathbf{a}_1 + \mathbf{a}_2, \quad \mathbf{u}_3 = \mathbf{a}_3, \quad \mathbf{u}_4 = i\xi\mathbf{a}_3 + \mathbf{a}_4\end{gathered}$$

$$\begin{gathered}\mathbf{u}_5 = \mathbf{a}_5, \quad \mathbf{u}_6 = \mathbf{a}_6, \quad \mathbf{u}_7 = \tfrac{1}{6}(i\xi)^3\mathbf{a}_1 + \tfrac{1}{2}(i\xi)^2\mathbf{a}_2 + i\xi\mathbf{a}_8 + \mathbf{a}_7 \\ \mathbf{u}_8 = \tfrac{1}{2}(i\xi)^2\mathbf{a}_1 + i\xi\mathbf{a}_2 + \mathbf{a}_8, \quad \mathbf{u}_9 = \tfrac{1}{6}(i\xi)^3\mathbf{a}_3 + \tfrac{1}{2}(i\xi)^2\mathbf{a}_4 + i\xi\mathbf{a}_{10} + \mathbf{a}_9 \\ \mathbf{u}_{10} = \tfrac{1}{2}(i\xi)^2\mathbf{a}_3 + i\xi\mathbf{a}_4 + \mathbf{a}_{10}, \quad \mathbf{u}_{11} = i\xi\mathbf{a}_5 + \mathbf{a}_{11}, \quad \mathbf{u}_{12} = i\xi\mathbf{a}_6 + \mathbf{a}_{12} \\ \boldsymbol{\sigma}_r = (0,0,0) \quad (r = \overline{1,6}) \\ \boldsymbol{\sigma}_7 = i\xi\mathbf{b}_8 + \mathbf{b}_7, \quad \boldsymbol{\sigma}_8 = \mathbf{b}_8, \quad \boldsymbol{\sigma}_9 = i\xi\mathbf{b}_{10} + \mathbf{b}_9 \\ \boldsymbol{\sigma}_{10} = \mathbf{b}_{10}, \quad \boldsymbol{\sigma}_{11} = \mathbf{b}_{11}, \quad \boldsymbol{\sigma}_{12} = \mathbf{b}_{12}\end{gathered} \tag{2.2}$$

In (2.1) $x_\alpha^* = h\xi_\alpha^*$ are the coordinates of the centre of twisting

$$\psi_{11} = -\psi_{22} = \nu / 2(\xi_2^2 - \xi_1^2), \quad \psi_{12} = \psi_{21} = -\nu\xi_1\xi_2$$
$$\tau_{\alpha\beta} = \partial_\beta\theta_\alpha + \psi_{\alpha\beta}, \quad \alpha, \beta = 1, 2$$

The functions θ_1, θ_2, θ, Φ solve the boundary-value problems

$$\begin{aligned} &\Delta\theta_\alpha = 2\xi_\alpha, \quad n_\beta\partial_\beta\theta_\alpha|_{\partial S} = n_\beta\psi_{\alpha\beta} \\ &\Delta\Phi = -2, \quad \partial_s\Phi|_{\partial S} = 0 \\ &\Delta\theta = 0, \quad n_\beta\partial_\beta\theta|_{\partial S} = \tfrac{1}{2}\partial_s(\xi_1^2 + \xi_2^2) \\ &E_0 = 2(1+\nu), \quad E = \mu E_0, \\ &\Delta = \partial_1^2 + \partial_2^2, \quad \partial_\alpha = \partial / \partial\xi_\alpha, \quad \partial_s = \partial / \partial s \end{aligned} \tag{2.3}$$

where μ is the shear modulus, ν is Poisson's ration, n_β is the projection of the unit normal vector on to the lateral surface of the cylinder, and s is a variable along the contour ∂S.

All the expressions above are written in dimensionless coordinates $\xi_\alpha = h^{-1}x_\alpha$, where h is a characteristic linear dimension of S. The displacement and stress vectors are also dimensionless and can be obtained by dividing the corresponding dimensional quantities by μ and h.

Because the static problem is considered, when constructing the vector-valued Green's function there is no need to require that $\mathbf{G}_0^+$ and $\mathbf{G}_0^-$ should satisfy the radiation condition, so that one does not have to change from the system of elementary solutions (2.2) to the fundamental system of elementary solutions [2]. $\mathbf{G}^+$ and $\mathbf{G}^-$ can be constructed using the following properties.

Property 1. A displacement of the cylinder as a rigid body corresponds to the system of elementary solutions (2.2) for $r = \overline{1, 6}$.

The following properties follow from the results of [2, 9, 10].

Property 2. The system of vectors $M = \{\mathbf{v}_r, \mathbf{v}_k^+, \mathbf{v}_k^-\}$ $(r = \overline{1, 12})$ is minimal and complete in H'.

Property 3. The systems of vectors $M_a^\pm = \{\mathbf{a}_r, \mathbf{a}_k^\pm\}$ $(r = \overline{1, 6})$ and $M_b^\pm = \{\mathbf{b}_r, \mathbf{b}_k^\pm\}$ $(r = \overline{7, 12})$ are minimal and complete in H.

Property 4. The following generalized orthogonality relations hold

$$[\mathbf{v}_{6+r}, \mathbf{v}_t] = -i(\mathbf{b}_{6+r}, \mathbf{a}_t)_H = p_r\delta_{rt}, \quad r, t = \overline{1, 6} \tag{2.4}$$
$$[\mathbf{v}_k^\pm, \mathbf{v}_t] = -i(\mathbf{b}_k^\pm, \mathbf{a}_t)_H = 0, \quad t = \overline{1, 6}$$

Here $p_1 = p_2 = D_1$, $p_3 = p_4 = D_2$, $p_5 = D_p$, $p_6 = D_{kp}$, $D_1 = h^{-4}E_0I_2$, $D_2 = h^{-4}E_0I_1$, $D_p = h^{-2}E_0|S|$, $D_{kp} = h^{-4}C$, I_1 and I_2 are the principal moments of inertia, $|S|$ is the area of cross-section, and C is the geometric twisting stiffness of the cross-section.

We shall now construct the vector-valued Green's function. Properties 1 and 2 enable us to seek it in the form (1.9) with

$$\mathbf{G}_0 = \sum_{r=7}^{12} C_r^\pm \mathbf{w}_r(\xi) \tag{2.5}$$

By Property 4, we have

$$C_r^\pm = \pm(2p_r)^{-1} i(\bar{\mathbf{a}}_r' \cdot \mathbf{P}), \quad r = \overline{7, 12} \tag{2.6}$$

$$C_k^\pm = \pm(2p_k^\pm)^{-1} i(\bar{\mathbf{a}}_k' \cdot \mathbf{P}) \tag{2.7}$$

$$\mathbf{P} = \mathbf{F} / \mu h^2, \quad \mathbf{a}_s' = \mathbf{a}_s(\xi_1', \xi_2')$$

Taking into account that $\mathbf{G}_0 = \{\mathbf{G}_{0u}, \mathbf{G}_{0\sigma}\}$, where $\mathbf{G}_{0u}$ and $\mathbf{G}_{0\sigma}$ are the u- and σ-components, by (2.5) and (2.6) we obtain the expressions for the displacement tensor and the force tensor

$$\mathbf{G}_{0u}^\pm = \mathbf{U}_{0u}^\pm \cdot \mathbf{e}, \quad \mathbf{G}_{0\sigma}^\pm = \mathbf{U}_{0\sigma}^\pm \cdot \mathbf{e} \tag{2.8}$$

$$\mathbf{U}_{0u}^{\pm} = \pm i \sum_{r=7}^{12} (2p_{r-6})^{-1} \mathbf{u}_r \bar{\mathbf{a}}'_r, \quad \mathbf{U}_{0\sigma}^{\pm} = \pm i \sum_{r=7}^{12} (2p_{r-6})^{-1} \boldsymbol{\sigma}_r \bar{\mathbf{a}}'_r$$

Naturally, the solution constructed has no independent physical meaning. It must therefore be considered as a partial solution of the inhomogeneous problem (1.1), which forms a part of the vector-valued Green's function (Green's tensor) for various boundary-value problems for a cylinder of finite length.

3. As an example, we consider the boundary-value problem for a cylinder of finite length (x, $[0, L]$), one end of which is clamped, the other one being stress-free. The cylinder undergoes a deformation under a point force applied at $O_1(x'_1, x'_2, x')$. Changing to the dimensionless coordinates, we shall seek a solution in the form

$$\begin{gathered} \mathbf{w}_G(\xi, \xi') = \mathbf{w}_0(\xi) + \mathbf{w}_1(\xi) + \mathbf{G}_0(\xi - \xi') + \mathbf{G}_1(\xi - \xi') \\ \mathbf{w}_0(\xi) = \sum_{r=1}^{12} A_r \mathbf{w}_r(\xi) \\ \mathbf{w}_1(\xi) = \sum_k [A_k^+ \mathbf{w}_k^+(\xi) + A_k^- \mathbf{w}_k^-(\xi - l)], \quad l = L/h \end{gathered} \tag{3.1}$$

We shall determine the coefficients of the expansion (3.1) from the boundary conditions

$$\mathbf{u}(0) = 0, \quad \boldsymbol{\sigma}(l) = 0 \tag{3.2}$$

Substituting (3.1) into (3.2), we obtain

$$\sum_{r=1}^{12} A_r \mathbf{a}_r + \sum_k [A_k^+ \mathbf{a}_k^+ + A_k^- e^{-i\bar{\alpha}_k l} \mathbf{a}_k^-] = -\mathbf{G}_u^-(-\xi') \tag{3.3}$$

$$\sum_{r=7}^{12} A_r \boldsymbol{\sigma}_r(l) + \sum_k [A_k^+ e^{i\alpha_k l} \mathbf{b}_k^+ + A_k^- \mathbf{b}_k^-] = -\mathbf{G}_\sigma^+(l - \xi') \tag{3.4}$$

$$\alpha_k = h\gamma_k, \quad \alpha_k = \alpha_k^+$$

The functional relationships (3.3) and (3.4) can be reduced to algebraic ones if Property 3 is used.

Thus, taking the scalar products of (3.4) by the elements of the subset $\{\mathbf{a}_r\}_{r=1}^{6}$ and using (2.4) and (2.5), one can determine the coefficients A_r, $r = \overline{7, 12}$. We have

$$\begin{gathered} A_7 = -C_7^+, \quad A_8 = i\xi' C_7^+ - C_8^+, \quad A_9 = -C_9^+, \\ A_{10} = i\xi' C_9^+ - C_{10}^+, \quad A_{11} = -C_{11}^+, \quad A_{12} = -C_{12}^+ \end{gathered} \tag{3.5}$$

The computation of the remaining coefficients of the expansion can be reduced to the solution of an infinite system of algebraic equations of the form

$$\begin{gathered} \sum_k c'_{km} e^{i\alpha_k l} A_k^+ + \sum_k c_{km} A_k^- = d_{m1} \\ -\sum_k \bar{c}_{mk} A_k^+ + \sum_k c'_{mk} e^{-i\bar{\alpha}_k l} A_k^- = d_{m2} \end{gathered} \tag{3.6}$$

$$ip_t A_t + \sum_k (c_{kt}^+ A_k^+ + c_{kt}^- e^{-i\bar{\alpha}_k l} A_k^-) = d_t \tag{3.7}$$

$$t = \overline{1, 6}$$

Here

$$d_{m1} = -\sum_k c'_{km} P_k e^{i\alpha_k (l - \xi')}, \quad P_k = i(2p_k^+)^{-1} (\bar{\mathbf{a}}_k^- \cdot \mathbf{P})$$

$$d_{m2} = -\sum_{r=7}^{12} c_{rm}^+ A_r - (\mathbf{b}_m^+, \mathbf{G}_u(-\xi'))_H$$

$$d_t = \sum_{r=7}^{12} \varphi_{rt} C_r^+ + \sum_k c_{kt}^- e^{-i\bar{\alpha}_k \xi'} C_k^-$$

$$c_{km} = (\mathbf{b}_k^-, \mathbf{a}_m^-)_H, \quad c'_{km} = (\mathbf{b}_k^+, \mathbf{a}_m^-)_H, \quad c_{kt}^\pm = (\mathbf{b}_{6+t}, \mathbf{a}_k^\pm)_H$$

$$c_{rm}^+ = (\mathbf{b}_m^+, \mathbf{a}_r)_H, \quad c_{rt} = (\mathbf{b}_{6+t}, \mathbf{a}_r)_H$$

$$\varphi_{rt} = -c_{rt} + (\mathbf{b}_{6+t}, \mathbf{u}_r(-\xi'))_H + i\xi' c_{rt}, \quad r = 7, 9$$

$$\varphi_{rt} = -c_{rt} + (\mathbf{b}_{6+t}, \mathbf{u}_r(-\xi'))_H, \quad r = 8, 10, 11, 12$$

When $\varepsilon = l^{-1}$ is a small parameter, the above relationships enable us to carry out an asymptotic analysis of various boundary-value problems and obtain accurate asymptotic estimates for the various components of the stress–strain state.

This will be demonstrated using the classical Saint-Venant problems as an example. To do this we set

$$\boldsymbol{\sigma}(l) = \boldsymbol{\sigma}^*(\xi_1, \xi_2), \quad \xi' = l$$

in the second boundary condition (3.2), and we introduce a new variable $\xi = l\zeta$. Applying the superposition principle, by (2.1), (2.6) and (2.7) we find that

$$C_k^\pm = \pm i(2p_k^\pm)^{-1} \int_S (\boldsymbol{\sigma}^* \cdot \bar{\mathbf{a}}_k^\mp) d\xi'_1 d\xi'_2$$

$$C_7^+ = \frac{iF_1^*}{2\mu h^2 D_1}, \quad C_8^+ = -\frac{M_1^*}{2\mu h^3 D_1}$$

$$C_9^+ = \frac{iF_2^*}{2\mu h^2 D_2}, \quad C_{10}^+ = \frac{M_2^*}{2\mu h^3 D_2} \tag{3.8}$$

$$C_{11}^+ = \frac{iF_3^*}{2\mu h^2 D_p}, \quad C_{12}^+ = \frac{iM_3^* + i(F_1^* \xi_2^* - F_2^* \xi_1^*)}{2\mu h^3 D_{kp}}$$

Here F_k, M_k^* are the projections of the principal vector and principal moment applied to the end $\xi = l$ of the cylinder.

From (3.5) and (3.8) one can see that A_r, $r = \overline{7, 12}$ can be determined exactly.

Turning to the infinite systems (3.6), in which we neglect the sums containing exponential factors, we can conclude that A_k^- are of order one in ε and

$$A_k^+ = \varepsilon^{-1} B_k^{(0)} + B_k^{(1)} + O(e^{-\beta_* l}), \quad \beta_* = \inf_k(\operatorname{Im} \alpha_k) \tag{3.9}$$

$$B_k^{(0)} = \varphi_{7k} C_7^+ + \varphi_{9k} C_9^+, \quad B_k^{(1)} = \sum_{\substack{r=8 \\ r \neq 9}}^{12} \varphi_{rk} C_r^+$$

Substituting (3.12) into (3.8), we get

$$A_t = \sum_{r=7}^{12} (\varphi_{rt} + \varphi'_{rt}) C_t^+ \tag{3.10}$$

$$\varphi'_{\alpha t} = \varepsilon^{-1} \sum_k c_{kt}^+ \varphi_{\alpha k}, \quad \alpha = 7, 9; \quad \varphi'_{st} = \sum_k c_{kt}^+ \varphi_{sk}, \quad s = 8, 10, 11, 12$$

Using (3.10), one can now analyse the effect of the exponential solutions (the boundary layer) on the inner deformed state through the constants A_t, ($t = \overline{1, 6}$) (they have no effect on the inner stress state because the stresses corresponding to the elementary solutions containing these coefficients are equal to zero). The leading terms of A_1 and A_3 are of order ε^{-3}, the correction term related to $\varphi'_{\alpha t}$ ($\alpha = 7, 9$) being of order ε^{-1}. The leading terms of A_2 and A_4 are of order ε^{-2}, the corrections being of order one. The leading terms of A_5 and A_6 are of order ε^{-1}, the correction terms being of order one.

The above analysis enables us to obtain accurate asymptotic estimates for the differences $\upsilon_i(\zeta) = u_i(\zeta) - u_i^0(\zeta)$, $\upsilon'_i = u_i - u_i - u_i^{0'}$, and $\tau_{ij} = \sigma_{ij} - \sigma_{ij}^0$, where u_i and σ_{ij} are the components of the displacement vector and the stress tensor of the exact solution, u_i^0 being the components of the displacement vector of Saint-Venant's theory in the case when

$$A_t = \sum_{r=7}^{12} C_r^+ \varphi_{rt} \tag{3.11}$$

where A_t are determined ignoring the boundary layer; $u_i^{0'}$ are the components of the displacement vector in the case when the A_t are determined from (3.10) (with the boundary layers taken into account).

In the case when $F_\alpha^* \neq 0$ $(\alpha = 1, 2)$, u_α has the leading order ε^{-3} among the displacement components, and σ_{33} has the leading order ε^{-1} among the stress components, all the remaining stresses being of order one, with

$$\varepsilon\upsilon_i = O(1), \quad \varepsilon\upsilon'_i = O(\chi), \quad \varepsilon\tau_{ij} = O(\chi) \tag{3.12}$$

In the case when $F_\alpha^* = 0$, $M_1^* \neq 0$, and $M_2^* = 0$ the component u_2 is of leading order ε^{-2}, all the stresses being of order one with

$$\upsilon_i = O(1), \quad \upsilon'_i = O(\chi), \quad \tau_{ij} = O(\chi) \tag{3.13}$$

In the case when $F_\alpha^* = 0$, $F_3^* \neq 0$ and $M_i^* \neq 0$, u_3 is the displacement of leading order ε^{-2}, all the stresses are of order one, and the estimates (3.13) are satisfied.

When $F_i^* = 0$, $M_\alpha^* = 0$, and $M_3^* \neq 0$, the displacements u_α have the leading order ε^{-2}, all the stresses are of order one, and the estimates (3.13) are satisfied.

The estimates are given for $0 < \zeta < 1$; $\chi = \exp(-\beta_* l\zeta)$ when $0 < \zeta < 1/2$, and $\chi = \varepsilon \exp(-\beta_* l(1 - \zeta))$ when $1/2 < \zeta < 1$.

Thus, the boundary layers enable us to obtain better estimates for the displacements in Saint-Venant's problems. They have no effect on the estimates for the stresses.

Here one should note that, first, the estimates obtained from τ_{ij} strongly support the Saint-Venant principle (a rigorous proof of which was first given by Toupin in [11]; profound extensions of Toupin's approach were obtained in [12]) and, secondly, no estimates for υ'_i can be established.

Using the superposition principle and the relationships obtained above, a similar asymptotic analysis can also be carried out in the case of arbitrary distributed external volume and surface forces.

We shall now define the vector-valued Green's function of Saint-Venant's theory for the boundary-value problem under consideration as

$$\mathbf{w}_G = \mathbf{G}_0 + \mathbf{w}_0 = \sum_{r=7}^{12} C_r^+ \mathbf{g}_r(\xi, \xi') \tag{3.14}$$

where C_r^+ are defined by (2.6) and

$$\mathbf{g}_r = \mathbf{g}_r^+, \quad 0 \leqslant \xi \leqslant \xi'; \qquad \mathbf{g}_r = \mathbf{g}_r^-, \quad \xi < \xi \leqslant l$$

$$\mathbf{g}_r^\pm = \delta_{s,r+1} i\xi' \mathbf{w}_r(\xi) + \sum_{t=1}^{6} \varphi_{rt} \mathbf{w}_t(\xi) + \mathbf{w}_r(\xi - \xi')$$

Using (3.14) we can obtain in an obvious way expressions for the displacement and force tensors. We have

$$\mathbf{U}_u^0 = \sum_{r=7}^{12} g_{ru} \bar{\mathbf{a}}_r, \quad \mathbf{U}_\sigma^0 = \sum_{r=7}^{12} g_{r\sigma} \bar{\mathbf{a}}_r \tag{3.15}$$

where g_{ru} and $g_{r\sigma}$ are the u- and σ-components of $\mathbf{g}_r$.

If, in place of a point force $\mathbf{F}$ applied at $O_1(x_1', x_2', x')$, we introduce an equivalent system consisting of a force $\mathbf{F}$ and a moment $\mathbf{M}$ applied at $O_2(0, 0, x')$, then (3.14) can be considered as a solution of the boundary-value problem within the framework of an applied theory taking shear into account. The constants C_r^+ must then be computed from (3.8) with $F_i^* = F_i$ and $M_i^* = M_i$. Many papers have been devoted to the construction of various versions of theories of this kind, a detailed survey of which can be found in [13, 14].†

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†See also: ELISEYEV V. V., One- and three-dimensional models in rigid rod mechanics. Doctorate dissertation, Leningrad, 1991.